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New results on holographic three-point functions

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Abstract: We exploit a gauge invariant approach for the analysis of the equations governing the dynamics of active scalar fluctuations coupled to the fluctuations of the metric along holographic RG flows. In the present approach, a second order ODE for the active scalar emerges rather simply and makes it possible to use the Green's function method to deal with (quadratic) interaction terms. We thus fill a gap for active scalar operators, whose three-point functions have been inaccessible so far, and derive a general, explicitly Bose symmetric formula thereof. As an application we compute the relevant three-point function along the GPPZ flow and extract the irreducible trilinear couplings of the corresponding superglueballs by amputating the external legs on-shell.

Keywords: Renormalization Group, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The holographic calculation of correlation functions in conformal field theories has been pursued much in recent years, spawned by the formulation of the AdS/CFT correspondence [1, 2]. Roughly speaking, the on-shell action of a (super)gravity theory living in a $(d+1)$-dimensional bulk space-time can be interpreted as the generating functional of a dual quantum field theory (QFT), if the bulk geometry is asymptotically anti-de Sitter (AdS). The prescribed boundary values of the bulk fields play the role of the QFT sources. If the bulk geometry is entirely AdS, the dual QFT is conformal. Otherwise, the bulk geometries describe renormalization group (RG) flows of the QFT away from the ultraviolet conformal fixed point, which are driven by relevant operators, either by deforming the action or by turning on non-zero vacuum expectation values (vevs). These are called operator and vev RG flows, respectively. The correspondence formula has been given a precise meaning by the development of a systematic renormalization procedure, known as holographic renormalization, that removes all infrared divergences stemming from the infinite volume in the bulk on-shell action and ensures the validity of the (anomalous) Ward identities in the boundary QFT [3]–[7].

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The most famous duality starts with gauged supergravity in $(d+1) = 5$ dimensions, which is obtained from $D = 10$ type-IIB supergravity by compactification on a five-sphere. Its AdS_5 solution is dual to $d = 4$, $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in the planar limit at strong 't Hooft coupling [8]. Several RG flows of this theory, with various degrees of supersymmetry, have been identified [9]–[18].

Explicit calculations of n-point functions in RG flows with $n > 2$ have been hampered by a number of reasons. First, in RG flow backgrounds, the fluctuations of the active scalar[1] couple to the fluctuations of the bulk metric in a non-trivial fashion even at the linearized level, and much effort was needed in previous work to disentangle this mixing and to extract some two-point functions [19]–[23]. Second, for many cases the background is such that the equations of motion are not explicitly solvable even for the simplest fields, i.e. inert scalars and transverse modes of the graviton and vector bosons. The favourite solvable cases are the GPPZ flow [16], which captures some features of the flow from $\mathcal{N} = 4$ to $\mathcal{N} = 1$ super Yang-Mills (SYM) theory, and the Coulomb branch flow [14], which breaks SU(4) R-symmetry to SO(4). The difficulties make approaching the calculation of higher-point functions rather daunting. However, interesting mass spectra have been exposed by two-point functions, and the explicit knowledge of three-point vertices would be very desirable. The first calculation of some very simple three-point functions of operators dual to inert scalars has been presented in [24].

In this paper, we consider the active scalar more thoroughly with the aim of providing a general and simple formula for the three-point function of its dual operator. Thanks to a Ward identity, this operator is equivalent to the trace of the QFT energy-momentum tensor in the case of operator RG flows. We shall apply the result to the GPPZ flow, which is an operator flow driven by the insertion of an operator of UV dimension $\Delta = 3$.

Tackling Einstein's equations to quadratic order around the RG flow background is quite an enterprise. Choosing a gauge to remove some of the fields is not of much help, since one cannot simply guess which gauge leads to the simplest equations for the remaining fields. Therefore, we develop and use a gauge invariant method. The idea is the following. From the various fields one can form a number of independent gauge invariant combinations (call them collectively I), which represent the true degrees of freedom. Using diffeomorphism invariance, Einstein's equations can then be equivalently expressed in terms of I only. This means that in the expansion to second order those fields that represent gauge artifacts can be dropped. Although this seems a lot like choosing a gauge, the equations to be solved are, in fact, gauge independent. Much to our surprise, this procedure leads to a second order ODE for the active scalar without much effort. Hence, interactions can be dealt with in the standard fashion by using the Green's function, and this makes three-point functions in principle accessible. It is, of course, impossible in most cases to perform the final bulk integral involving three bulk-to-boundary propagators. Nevertheless one can extract irreducible vertices by amputating on-shell external legs.

[1] In the common nomenclature, an *active* scalar is dual to the operator driving the RG flow and has a non-zero background, whereas the other scalars are called *inert*.

To finish the introduction, let us briefly introduce the problem and outline the rest of the paper. Our task will be to analyze the dynamics of bulk gravity coupled to a scalar field. Einstein's equations can be conveniently cast into the form[2]

$$E_{\mu\nu} = \tilde{R}_{\mu\nu} + 2S_{\mu\nu} = 0\,, \tag{1.1}$$

where

$$S_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \frac{2}{d-1}\tilde{g}_{\mu\nu}V(\phi)\,. \tag{1.2}$$

Bianchi's identity implies the equation of motion for the scalar,

$$\tilde{\nabla}^2\phi - \frac{\partial V}{\partial\phi} = 0\,. \tag{1.3}$$

We analyze the equations of motion using the time slicing formalism, in which the bulk metric takes the form

$$\mathrm{d}s^2 = (n^2 + n_i n^i)\mathrm{d}r^2 + 2n_i \mathrm{d}r\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{1.4}$$

A brief summary of the time slicing formalism is presented in section 2, which can be skipped in a cursory reading. The time slicing formalism has the virtue that the equations are expressed covariantly in terms of hyper surface quantities, which makes the calculations less prone to errors.

We wish to consider fluctuations around a supergravity background that is dual to an RG flow. Therefore, we expand the fields as follows,

$$\begin{aligned}
\phi &= \bar{\phi}(r) + \varphi\,,\\
g_{ij} &= \mathrm{e}^{2A(r)}\left(\eta_{ij} + h_{ij}\right),\\
n_i &= \nu_i\,,\\
n &= 1 + \nu\,,
\end{aligned} \tag{1.5}$$

where φ, h_{ij}, ν_i an ν denote the small fluctuations. The background functions $\bar{\phi}(r)$ and $A(r)$ satisfy gradient flow equations governed by a superpotential $W(\phi)$ according to [11]

$$\begin{aligned}
\partial_r\bar{\phi}(r) &= W_\phi\,,\\
\partial_r A(r) &= -\frac{2}{d-1}W\,,\\
\frac{1}{2}W_\phi^2 - \frac{d}{d-1}W^2 &= V\,.
\end{aligned} \tag{1.6}$$

Here and henceforth, we denote $W_\phi = \mathrm{d}W(\phi)/\mathrm{d}\phi|_{\bar{\phi}}$, and similar for higher derivatives. Furthermore, we adopt the convention that the indices of the fluctuations as well as of the derivatives ∂_i are raised and lowered with the flat (Minkowski/euclidean) metric.

[2]For our notation, see section 2.

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In section 3, we develop the gauge invariant approach that starts by identifying gauge invariant combinations, I, of the fluctuations φ, ν, ν^i and h^i_j. It is then demonstrated that Einstein's equations can be equivalently written in terms of I only, and a simple recipe is given how to achieve this goal. The gauge invariant field equations governing the dynamics of the active scalar are given explicitly in section 4. Two of the three coupled equations are easily solved, and the remaining equation for the active scalar is a linear second order ODE with higher order source terms. In section 5, we derive a general formula for the three-point function of the operator that is dual to the active scalar. It is given in terms of an integral over the radial bulk variable involving three bulk-to-boundary propagators and an operator involving the external momenta and radial derivatives. Some effort is needed to arrive at an expression that is explicitly Bose symmetric. The formula is applied to the GPPZ flow, in which case one can perform the integral, if all external legs are on-shell and amputated. Finally, section 6 contains our conclusions. Useful details for the GPPZ flow are listed in appendix A.

2. Geometric relations for hyper surfaces

The time slicing (or ADM) formalism [25, 26], which we will employ in our analysis of Einstein's equations, makes essential use of the geometry of hyper surfaces [27]. Therefore, we shall begin with a review of the basic relations governing the geometry of hyper surfaces.

A hyper surface in a space-time with coordinates X^μ ($\mu = 0, \ldots, d$) and metric $\tilde{g}_{\mu\nu}$ is defined by a set of $d+1$ functions, $X^\mu(x^i)$ ($i = 1, \ldots, d$), where the x^i are a set of (local) coordinates on the hyper surface. The tangents, $X^\mu_i \equiv \partial_i X^\mu$, and the normal vector, N^μ, to the hyper surface satisfy the following orthogonality relations,

$$\begin{aligned} \tilde{g}_{\mu\nu} X^\mu_i X^\nu_j &= g_{ij} \,, \\ \tilde{g}_{\mu\nu} X^\mu_i N^\nu &= 0 \,, \\ \tilde{g}_{\mu\nu} N^\mu N^\nu &= 1 \,, \end{aligned} \tag{2.1}$$

where g_{ij} represents the (induced) metric on the hyper surface. Henceforth, a tilde will be used to label quantities characterizing the $(d+1)$-dimensional space-time manifold, whereas those of the hyper surface remain unadorned.

The equations of Gauss and Weingarten define the second fundamental form, $\mathcal{K}_{ij}$, of the hyper surface,

$$\partial_i X^\mu_j + \tilde{\Gamma}^\mu{}_{\lambda\nu} X^\lambda_i X^\nu_j - \Gamma^k{}_{ij} X^\mu_k = \mathcal{K}_{ij} N^\mu \,, \tag{2.2}$$

$$\partial_i N^\mu + \tilde{\Gamma}^\mu{}_{\lambda\nu} X^\lambda_i N^\nu = -\mathcal{K}^j_i X^\mu_j \,. \tag{2.3}$$

The second fundamental form describes the extrinsic curvature of the hyper surface and is related to the intrinsic curvature by another equation of Gauss,[3]

$$\tilde{R}_{\mu\nu\lambda\rho} X^\mu_i X^\nu_j X^\lambda_k X^\rho_l = R_{ijkl} - \mathcal{K}_{il}\mathcal{K}_{jk} + \mathcal{K}_{ik}\mathcal{K}_{jl} \,. \tag{2.4}$$

[3]Our convention for the Riemann tensor is $R^i{}_{jkl} = \partial_l \Gamma^i{}_{jk} + \Gamma^i{}_{lm}\Gamma^m{}_{jk} - (k \leftrightarrow l)$.

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Furthermore, it satisfies the equation of Codazzi,

$$\tilde{R}_{\mu\nu\lambda\rho} X_i^\mu X_j^\nu N^\lambda X_k^\rho = \nabla_j \mathcal{K}_{ik} - \nabla_i \mathcal{K}_{jk} \,. \tag{2.5}$$

The symbol ∇ denotes covariant derivatives with respect to the induced metric, g_{ij}.

The above formulae simplify, if (as in the familiar time slicing formalism), we choose space-time coordinates such that

$$X^0 = const \,, \qquad X^i = x^i \,. \tag{2.6}$$

Then, the tangent vectors are given by $X_i^0 = 0$ and $X_i^j = \delta_i^j$. One conveniently splits up the space-time metric as (shown here for euclidean signature)

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} n_i n^i + n^2 & n_j \\ n_i & g_{ij} \end{pmatrix} , \tag{2.7}$$

whose inverse is given by

$$\tilde{g}^{\mu\nu} = \frac{1}{n^2} \begin{pmatrix} 1 & -n^j \\ -n^i & n^2 g^{ij} + n^i n^j \end{pmatrix} . \tag{2.8}$$

The matrix g^{ij} is the inverse of g_{ij} and is used to raise hyper surface indices. The quantities n and n^i are the lapse function and shift vector, respectively.

The normal vector N^μ satisfying the orthogonality relations (2.1) is given by

$$N_\mu = (n, 0) \,, \qquad N^\mu = \frac{1}{n}(1, -n^i) \,. \tag{2.9}$$

Then, one can obtain the second fundamental form from the equation of Gauss (2.2) as

$$\mathcal{K}_{ij} = -\frac{1}{2n} \left(\partial_0 g_{ij} - \nabla_i n_j - \nabla_j n_i \right) . \tag{2.10}$$

We are interested in expressing all bulk quantities in terms of hyper surface quantities. Using the equations of Gauss and Weingarten, some Christoffel symbols can be expressed as follows,

$$\tilde{\Gamma}^0{}_{ij} = \frac{1}{n} \mathcal{K}_{ij} \,, \tag{2.11}$$

$$\tilde{\Gamma}^k{}_{ij} = \Gamma^k{}_{ij} - \frac{n^k}{n} \mathcal{K}_{ij} \,, \tag{2.12}$$

$$\tilde{\Gamma}^0{}_{i0} = \frac{1}{n} \partial_i n + \frac{n^j}{n} \mathcal{K}_{ij} \,, \tag{2.13}$$

$$\tilde{\Gamma}^k{}_{i0} = \nabla_i n^k - \frac{n^k}{n} \partial_i n - n \mathcal{K}_{ij} \left(g^{jk} + \frac{n^j n^k}{n^2} \right) . \tag{2.14}$$

The remaining components, $\tilde{\Gamma}^0{}_{00}$ and $\tilde{\Gamma}^k{}_{00}$, are easily found from their definitions using (2.7) and (2.8),

$$\tilde{\Gamma}^0{}_{00} = \frac{1}{n} \left(\partial_0 n + n^j \partial_j n + n^i n^j \mathcal{K}_{ij} \right) \,, \tag{2.15}$$

$$\tilde{\Gamma}^k{}_{00} = \partial_0 n^k + n^i \nabla_i n^k - n \nabla^k n - 2n \mathcal{K}_i^k n^i - \frac{n^k}{n} \left(\partial_0 n + n^j \partial_j n + n^i n^j \mathcal{K}_{ij} \right) . \tag{2.16}$$

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3. Gauge invariant approach

3.1 Gauge transformations and invariants

Invariance under diffeomorphisms is a powerful tool that allows one to reduce the number of fields in the field equations to the effective degrees of freedom, which is usually done by fixing a gauge. In our treatment of the coupled scalar-gravity system we shall take a slightly different approach identifying gauge invariant quantities, in terms of which the field equations are expressed. Thereby we obtain the equations of motion in a gauge invariant form. Alternatively, our treatment indicates a choice of gauge, in which the equations of motion become particularly simple.

Let us start with the transformations under a diffeomorphism of the form

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)\,, \tag{3.1}$$

where ξ is infinitesimal. Because general covariance implies that Einstein's equations are invariant under any diffeomorphism, they are invariant under (3.1) to any order in ξ, and it is sufficient to consider the first order in ξ only. Under (3.1), a scalar field transforms as

$$\delta\phi = \xi^\mu \partial_\mu \phi\,, \tag{3.2}$$

whereas a covariant tensor of rank two (e.g. the metric or Einstein's equations) transforms as

$$\delta E_{\mu\nu} = \partial_\mu \xi^\lambda E_{\lambda\nu} + \partial_\nu \xi^\lambda E_{\mu\lambda} + \xi^\lambda \partial_\lambda E_{\mu\nu}\,. \tag{3.3}$$

Splitting the fields into background and fluctuations as in (1.5), the transformations (3.2) and (3.3) become gauge transformations for the fluctuations,

$$\begin{aligned}
\delta\varphi &= W_\phi \xi^r + \mathcal{O}(f)\,,\\
\delta\nu &= \partial_r \xi^r + \mathcal{O}(f)\,,\\
\delta\nu^i &= \partial^i \xi^r + \mathrm{e}^{2A} \partial_r \xi^i + \mathcal{O}(f)\,,\\
\delta h^i_j &= \partial_j \xi^i + \partial^i(\eta_{jk}\xi^k) - \frac{4}{d-1} W \delta^i_j \xi^r + \mathcal{O}(f)\,.
\end{aligned} \tag{3.4}$$

By $\mathcal{O}(f^n)$ we denote terms of order n in the fluctuations φ, h_{ij}, ν_i and ν. (Remember that we are always at first order in the gauge parameter ξ.) Moreover, we split the metric fluctuations h^i_j as follows,

$$h^i_j = {h^{TT}}^i_j + \partial^i \epsilon_j + \partial_j \epsilon^i + \frac{\partial^i \partial_j}{\Box} H + \frac{1}{d-1} \delta^i_j h\,, \tag{3.5}$$

where ${h^{TT}}^i_j$ denotes the traceless transversal part, and ϵ^i is a transversal vector ($\partial_i \epsilon^i = 0$). It is straightforward to obtain from (3.4) the transformation laws for these components,

$$\begin{aligned}
\delta {h^{TT}}^i_j &= \mathcal{O}(f)\,,\\
\delta\epsilon^i &= \Pi^i_j \xi^j + \mathcal{O}(f)\,,\\
\delta H &= 2\partial_i \xi^i + \mathcal{O}(f)\,,\\
\delta h &= -4W\xi^r + \mathcal{O}(f)\,.
\end{aligned} \tag{3.6}$$

Here, Π^i_j is the transversal projector,

$$\Pi^i_j = \delta^i_j - \frac{\partial^i \partial_j}{\Box} \,. \tag{3.7}$$

Using the transformation laws (3.4) and (3.6) we can construct the following gauge invariant combinations of the fluctuations,

$$\begin{aligned}
a &= \varphi + W_\phi \frac{h}{4W} + \mathcal{O}(f^2)\,, && (3.8)\\
b &= \nu + \partial_r \left(\frac{h}{4W}\right) + \mathcal{O}(f^2)\,, && (3.9)\\
c &= \partial_i \nu^i + \Box \frac{h}{4W} - \frac{1}{2} \mathrm{e}^{2A} \partial_r H + \mathcal{O}(f^2)\,, && (3.10)\\
d^i &= \Pi^i_j \nu^j - \mathrm{e}^{2A} \partial_r \epsilon^i + \mathcal{O}(f^2)\,, && (3.11)\\
e^i_j &= h^{TT}{}^i_j + \mathcal{O}(f^2)\,. && (3.12)
\end{aligned}$$

Here, c and d^i both stem from $\delta\nu^i$, which we have split into its longitudinal and transversal parts. It is in principle possible to find the higher order terms in (3.8)–(3.12) explicitly, but we shall argue in the next subsection that this is not necessary for our purposes.

We would like to finish this subsection by making two crucial observations. For convenience, let us arrange the fluctuations into two sets, $X = (h, H, \epsilon^i)$, and $Y = (\varphi, \nu, \nu^i, h^{TT}{}^i_j)$. Furthermore, let us collect also the invariants to $I = (a, b, c, d^i, e^i_j)$. We shall henceforth use X, Y, and I to denote any of the fields of the corresponding set. The first observation is that we can write the gauge parameter ξ to first order as a linear functional of the variations of the fields X,

$$\xi^\lambda = z^\lambda(\delta X) + \mathcal{O}(f^2) = \delta z^\lambda(X) + \mathcal{O}(f^2)\,. \tag{3.13}$$

This is clear from (3.6). Second, the gauge invariant combinations have been chosen such that the fields Y can be written in the form

$$Y = I + y(X) + \mathcal{O}(f^2)\,, \tag{3.14}$$

where y is a linear functional of the fields X. Moreover, when going to the next order in the fluctuations, one can choose I such that the quadratic terms do not contain terms with two Is, i.e.

$$Y = I + y(X) + \alpha(X, X) + \beta(X, I) + \mathcal{O}(f^3)\,, \tag{3.15}$$

where α and β are bilinear in their arguments. Let us now turn our attention to the field equations.

3.2 Einstein's equations and gauge invariance

It is our goal to re-write Einstein's equations to quadratic order in the fluctuations in terms of the gauge invariant combinations I. To do this, we start by expanding them symbolically in the form

$$E_{\mu\nu} = E^{(1)1}_{\mu\nu}(X) + E^{(1)2}_{\mu\nu}(Y) + E^{(2)1}_{\mu\nu}(X, X) + E^{(2)2}_{\mu\nu}(X, Y) + E^{(2)3}_{\mu\nu}(Y, Y) + \mathcal{O}(f^3)\,. \tag{3.16}$$

Here, $E^{(1)}$ and $E^{(2)}$ denote linear and bilinear terms, respectively. The background equations are satisfied identically. Now we substitute (3.15) for Y, which yields

$$E_{\mu\nu} = \tilde{E}^{(1)1}_{\mu\nu}(X) + E^{(1)2}_{\mu\nu}(I) + \tilde{E}^{(2)1}_{\mu\nu}(X,X) + \tilde{E}^{(2)2}_{\mu\nu}(X,I) + E^{(2)3}_{\mu\nu}(I,I) + \mathcal{O}(f^3)\,. \tag{3.17}$$

Notice that the functionals $E^{(1)2}$ and $E^{(2)3}$ are essentially unchanged, whereas the others gets modified. For example, $\tilde{E}^{(2)2}$ receives contributions from $E^{(2)2}$, $E^{(2)3}$ and $E^{(1)2}$ [through β in (3.15)].

Eq. (3.17) can be vastly simplified by considering its transformation under diffeomorphisms. From (3.3) and (3.13) we find

$$\delta E_{\mu\nu} = \partial_\mu \delta z^\lambda(X) E_{\lambda\nu} + \partial_\nu \delta z^\lambda(X) E_{\mu\lambda} + \delta z^\lambda(X)\partial_\lambda E_{\mu\nu} + \mathcal{O}(f^3)\,. \tag{3.18}$$

As there is no first order term on the right hand side, the first order terms of $E_{\mu\nu}$ must be invariant, which implies that $\tilde{E}^{(1)1}_{\mu\nu}(X)=0$ in (3.17). We have explicitly checked that this is the case. Then, substituting $E_{\mu\nu} = E^{(1)2}_{\mu\nu}(I) + \mathcal{O}(f^2)$ into the right hand side of (3.18) yields

$$\delta E_{\mu\nu} = \delta\left[\partial_\mu z^\lambda(X) E^{(1)2}_{\lambda\nu}(I) + \partial_\nu z^\lambda(X) E^{(1)2}_{\mu\lambda}(I) + z^\lambda(X)\partial_\lambda E^{(1)2}_{\mu\nu}(I)\right] + \mathcal{O}(f^3)\,. \tag{3.19}$$

Comparing (3.19) with (3.17), we find that

$$\tilde{E}^{(2)1}_{\mu\nu} = 0\,, \tag{3.20}$$

$$\tilde{E}^{(2)2}_{\mu\nu} = \partial_\mu z^\lambda(X) E^{(1)2}_{\lambda\nu}(I) + \partial_\nu z^\lambda(X) E^{(1)2}_{\mu\lambda}(I) + z^\lambda(X)\partial_\lambda E^{(1)2}_{\mu\nu}(I)\,. \tag{3.21}$$

Moreover, every single term in $\tilde{E}^{(2)2}_{\mu\nu}$ contains the first order equations of motion. Therefore, it can be dropped. (We can freely use the first order equations to simplify the interaction terms). Thus, we arrive at the following equation to be solved,

$$E^{(1)2}_{\mu\nu}(I) + E^{(2)3}_{\mu\nu}(I,I) = 0\,, \tag{3.22}$$

which is obtained by the following recipe. *We expand the equations of motion to second order in the fluctuations X and Y. Then, we replace every Y by its corresponding I and simply drop all Xs.* This simple rule is summarized by the following substitutions,

$$\phi \to a\,, \qquad \nu \to b\,, \qquad \nu^i \to d^i + \frac{\partial^i c}{\Box}\,, \qquad h^i_j \to e^i_j\,. \tag{3.23}$$

For the sake of completeness, we list here the expressions that follow from the rules (3.23) for the quantities that appear in the field equations. The extrinsic curvature becomes

$$\begin{aligned} n\mathcal{K}^i_j \to\ & \frac{2}{d-1} W\delta^i_j - \frac{1}{2}\partial_r e^i_j + \frac{1}{2}\mathrm{e}^{-2A}\left(\partial^i d_j + \partial_j d^i + 2\frac{\partial^i\partial_j}{\Box}c\right) + \frac{1}{2} e^i_k \partial_r e^k_j + \\ & -\frac{1}{2}\mathrm{e}^{-2A}\left[e^i_k\left(\partial^k d_j + \partial_j d^k + 2\frac{\partial_j\partial^k}{\Box}c\right) + \left(d^k + \frac{\partial^k c}{\Box}\right)\left(\partial^i e_{jk} + \partial_j e^i_k - \partial_k e^i_j\right)\right], \end{aligned} \tag{3.24}$$

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and its trace is

$$n\mathcal{K}^i_i \to \frac{2d}{d-1}W + \mathrm{e}^{-2A}c + \frac{1}{2}e^i_k\partial_r e^k_i - \mathrm{e}^{-2A}e^i_k\left(\partial^k d_i + \frac{\partial_i\partial^k}{\Box}c\right). \tag{3.25}$$

The intrinsic Ricci tensor is replaced by

$$\begin{aligned} R_{ij} \to\ & \frac{1}{2}\Box e_{ij} + \frac{1}{2}e^k_l\left(\partial_i\partial_k e^l_j + \partial_j\partial_k e^l_i - \partial_k\partial^l e_{ij} - \partial_i\partial_j e^l_k\right) - \\ & -\frac{1}{4}(\partial_i e^k_l)(\partial_j e^l_k) + \frac{1}{2}(\partial_l e^k_i)(\partial_k e^l_j) - \frac{1}{2}(\partial_l e^k_j)(\partial^l e_{ik}), \end{aligned} \tag{3.26}$$

and the Ricci scalar becomes

$$R \to -\mathrm{e}^{-2A}\left[e^i_j\Box e^j_i + \frac{3}{4}(\partial_i e^j_k)(\partial^i e^k_j) - \frac{1}{2}(\partial_i e^k_j)(\partial^j e^i_k)\right]. \tag{3.27}$$

We finish this subsection by making two remarks on our method. First, the substitutions (3.23) can also be interpreted as a choice of gauge, namely $X = 0$. Second, it is straightforward to extend the present analysis to the case with more than one scalar. For scalars ϕ^I that have a canonical kinetic term one can define the gauge invariants a^I by replacing in (3.8) W_ϕ by $W_I = \partial W/\partial\phi^I$. In particular, inert scalars are gauge invariant up to $\mathcal{O}(f)$.

4. Field Equations

Let us now consider in detail the equations governing the dynamics of the fluctuations of an active scalar. Active scalars mix with the metric fluctuations even at the linearized level, and a lot of effort was devoted in previous work to resolve this mixing [19]–[23]. As we shall see, in our gauge invariant approach, or, equivalently, in the gauge $X = 0$, this is much simpler. We will obtain a second order ODE for the active scalar fluctuation in a rather straightforward fashion.

We start with the equation of motion for the scalar,[4] (1.3), which takes the form

$$\begin{aligned} \Big[&\partial_r^2 - 2n^i\partial_r\partial_i + n^i n^j\nabla_i\partial_j + n^2\nabla^2 - \left(n\mathcal{K}^i_i + \partial_r\ln n - n^i\partial_i\ln n\right)\partial_r - \\ &- \left(\partial_r n^k - n^i\nabla_i n^k - n\nabla^k n - n^k\partial_r\ln n + n^k n^i\partial_i\ln n - n^k n\mathcal{K}^i_i\right)\partial_k\Big]\phi - n^2\frac{\partial V}{\partial\phi} = 0. \end{aligned} \tag{4.1}$$

We have used the expressions listed in section 2 for the bulk connections.

After expanding (4.1) to second order and using the substitution rule (3.23), we obtain

$$\left(\partial_r^2 - \frac{2d}{d-1}W\partial_r + \mathrm{e}^{-2A}\Box - V_{\phi\phi}\right)a - W_\phi\mathrm{e}^{-2A}c - W_\phi\partial_r b - 2V_\phi b = J_a, \tag{4.2}$$

[4]The equation for the scalar follows from Einstein's equations. However, we need not consider the components E_{ij}, if we are not interested in the fluctuations e^i_j.

where the quadratic source J_a is given by

$$
\begin{aligned}
J_a = {} & \frac{1}{2}V_{\phi\phi\phi}a^2 + V_\phi b^2 + 2V_{\phi\phi}ab - W_\phi b\partial_r b + (\partial_r a)(\partial_r b) + \frac{1}{2}W_\phi e^i_j \partial_r e^j_i + \\
& + e^{-2A}\Bigg[- 2b\Box a - (\partial^i b)(\partial_i a) + c\partial_r a + 2\left(d^i + \frac{\partial^i c}{\Box}\right)\partial_i\partial_r a - \\
& \quad - W_\phi\left(d^i + \frac{\partial^i c}{\Box}\right)\partial_i b + \left(\partial_r d^i + \partial_r\frac{\partial^i c}{\Box}\right)\partial_i a - 2\frac{d-2}{d-1}W\left(d^i + \frac{\partial^i c}{\Box}\right)\partial_i a + \\
& \quad + e^i_j\left(\partial_i\partial^j a - W_\phi\partial_i d^j - W_\phi\frac{\partial_i\partial^j}{\Box}c\right)\Bigg]\,.
\end{aligned}
\tag{4.3}
$$

Next, we turn to Einstein's equations, which we need to consider only the normal and mixed components of. These are easily obtained multiplying (1.1) by $N^\mu N^\nu - g^{ij}X^\mu_i X^\nu_j$ and $N^\mu X^\nu_i$, respectively, using the geometrical relations of section 2. For the normal components we find

$$
\begin{aligned}
& (n\mathcal{K}^i_j)(n\mathcal{K}^j_i) - (n\mathcal{K}^i_i)(n\mathcal{K}^j_j) - n^2R - 2n^2g^{ij}(\partial_i\phi)(\partial_j\phi) - 4n^2V + \\
& \qquad + 2(\partial_r\phi)^2 - 4n^i(\partial_i\phi)(\partial_r\phi) + 2n^in^j(\partial_i\phi)(\partial_j\phi) = 0\,.
\end{aligned}
\tag{4.4}
$$

Expanding to second order and replacing the fields by means of (3.23) we obtain

$$
-4We^{-2A}c + 4W_\phi\partial_r a - 4V_\phi a - 8Vb = J_c\,,
\tag{4.5}
$$

with the quadratic source term J_c given by

$$
\begin{aligned}
J_c = {} & 4Vb^2 + 8V_\phi ab + 2V_{\phi\phi}a^2 - 2(\partial_r a)^2 + (e^{-2A}c)^2 + 2e^{-2A}(\partial^i a)(\partial_i a) + \\
& + 4W_\phi e^{-2A}\left(d^i + \frac{\partial^i c}{\Box}\right)\partial_i a + 2We^i_j\partial_r e^j_i - 4We^{-2A}e^i_j\partial_i\left(d^j + \frac{\partial^j c}{\Box}\right) - \\
& - \frac{1}{4}(\partial_r e^i_j)(\partial_r e^j_i) + e^{-2A}\left(\partial_i d^j + \frac{\partial_i\partial^j}{\Box}c\right)\partial_r e^i_j - \\
& - e^{-4A}\left[\frac{1}{2}(\partial_i d^j)(\partial^i d_j) + \frac{1}{2}(\partial_i d^j)(\partial_j d^i) + 2(\partial_i d^j)\frac{\partial^i\partial_j}{\Box}c + \left(\frac{\partial_i\partial^j}{\Box}c\right)\left(\frac{\partial^i\partial_j}{\Box}c\right)\right] - \\
& - e^{-2A}\left[e^i_j\Box e^j_i + \frac{3}{4}(\partial_i e^j_k)(\partial^i e^k_j) - \frac{1}{2}(\partial_i e^j_k)(\partial^k e^i_j)\right]\,.
\end{aligned}
\tag{4.6}
$$

Similarly, the mixed components of Einstein's equations are rewritten as

$$
\partial_i(n\mathcal{K}^j_j) - \nabla_j(n\mathcal{K}^j_i) - (\partial_i\ln n)(n\mathcal{K}^j_j) + (\partial_j\ln n)(n\mathcal{K}^j_i) - 2(\partial_i\phi)(\partial_r\phi - n^j\partial_j\phi) = 0\,,
\tag{4.7}
$$

which yields the equation

$$
-\frac{1}{2}e^{-2A}\Box d_i - 2W\partial_i b - 2W_\phi\partial_i a = J_i\,.
\tag{4.8}
$$

where J_i to quadratic order is given by

$$\begin{aligned}
J_i =& -W\partial_i b^2 + 2(\partial_i a)(\partial_r a) + \mathrm{e}^{-2A}(\Pi_i^j c)(\partial_j b) + \frac{1}{2}(\partial_j b)(\partial_r e_i^j) - \frac{1}{2}\mathrm{e}^{-2A}\frac{\partial_j c}{\Box}\Box e_i^j - \\
& -\frac{1}{4}\partial_i\partial_r(e_k^j e_j^k) + \frac{1}{2}e_k^j\partial_r\partial_j e_i^k + \frac{1}{4}(\partial_i e_k^j)(\partial_r e_j^k) - \frac{1}{2}\mathrm{e}^{-2A}e_k^j\partial_j(\partial^k d_i - \partial_i d^k) - \\
& -\frac{1}{2}\mathrm{e}^{-2A}(\partial_j e_i^k)(\partial^j d_k - \partial_k d^j) - \frac{1}{2}\mathrm{e}^{-2A}d_j\Box e_i^j - \frac{1}{2}\mathrm{e}^{-2A}(\partial_j b)(\partial^j d_i + \partial_i d^j)\,.
\end{aligned} \tag{4.9}$$

Our strategy is to solve (4.5) and (4.8) for b, c and d_i and substitute them into (4.2) to obtain an equation for a, which is still coupled to e^i_j through the source terms. Thus, we find

$$\begin{aligned}
b &= -\frac{W_\phi^2}{W^2}\tilde{a} - \frac{1}{2W}\frac{\partial_i}{\Box}J^i\,, \\
\Box d_i &= -2\mathrm{e}^{2A}\Pi_i^j J_j\,, \\
\mathrm{e}^{-2A}c &= \frac{W_\phi^2}{W^2}\partial_r\tilde{a} - \frac{1}{4W}J_c + \frac{V}{W^2}\frac{\partial_i}{\Box}J^i\,,
\end{aligned} \tag{4.10}$$

where we have defined $\tilde{a} = (W/W_\phi)a$ for later convenience.

Substituting (4.10) into (4.2) yields

$$\left(D^2 + \mathrm{e}^{-2A}\Box\right)\tilde{a} = J_{\tilde{a}}\,, \tag{4.11}$$

where we have abbreviated

$$D^2 = \left[\partial_r + 2\left(W_{\phi\phi} - \frac{W_\phi^2}{W} - \frac{d}{d-1}W\right)\right]\partial_r\,, \tag{4.12}$$

and the source term is given by

$$J_{\tilde{a}} = \frac{W}{W_\phi}J_a - \frac{1}{4}J_c - \frac{1}{2}\left[\partial_r + 2\left(W_{\phi\phi} - \frac{W_\phi^2}{W} - \frac{d}{d-1}W\right)\right]\frac{\partial_i}{\Box}J^i\,. \tag{4.13}$$

As promised, we find that the equation for $\tilde{a}$ at the linearized level is rather simple compared to the effort needed in previous work. More importantly, since it is a second order ordinary differential equation (after going to momentum space), it is possible to use the standard Green's function method for going beyond the linearized level.

5. Correlation functions

5.1 General considerations

We shall now calculate correlation functions of the dual deformed conformal field theory. The presentation in this and the next subsection will be general, if not otherwise indicated. We start here with some general considerations including also the use of the Green's function method for calculating the interaction terms. We use the variable $\rho = \mathrm{e}^{-2r}$ in many formulae. In subsection 5.2 the calculation of the three-point function of the active scalar will be presented. Subsection 5.3 is dedicated entirely to the results for the GPPZ flow. As already mentioned, useful relations for the GPPZ flow are listed in appendix A.

The bulk fluctuations h^i_j and φ are the duals of the boundary energy momentum tensor, T^i_j, and the scalar operator $\mathcal{O}$ of conformal dimension Δ, respectively. This is made explicit through the couplings of their boundary values to these operators,

$$\int \mathrm{d}^d x \left(\frac{1}{2} \hat{h}^j_i T^i_j + \hat{\varphi} \mathcal{O} \right) , \tag{5.1}$$

where $\hat{h}^i_j$ and $\hat{\varphi}$ are defined as the leading coefficients in the asymptotic expansions

$$h^i_j(x,\rho) = \hat{h}^i_j(x) + \cdots + \check{h}^i_j(x)\rho^{d/2} + \cdots , \tag{5.2}$$

$$\varphi(x,\rho) = \hat{\varphi}(x)\rho^{(d-\Delta)/2} + \cdots + \check{\varphi}(x)\rho^{\Delta/2} + \cdots . \tag{5.3}$$

Moreover, $\check{h}^i_j$ and $\check{\varphi}$ are the leading coefficients of the sub-leading series and are called the responses [22]. We have not written the sub-leading terms, which can also include logarithms. The response functions determine the *exact* one-point functions of the corresponding dual operators. More precisely, in the transversal gauge, $\nu = \nu^i = 0$, the exact one-point function $\langle \mathcal{O} \rangle$ becomes [5, 6]

$$\langle \mathcal{O} \rangle = (2\Delta - d)\check{\varphi} + \text{contact terms} . \tag{5.4}$$

where the contact terms are finite but in principle scheme dependent. However, we can express $\langle \mathcal{O} \rangle$ in a manifestly gauge invariant form after observing that $\check{\varphi}$ is equal to $\check{a}$ plus a (gauge dependent) function of the sources, so that (5.4) simply becomes

$$\langle \mathcal{O} \rangle = (2\Delta - d)\check{a} + \text{contact terms} . \tag{5.5}$$

We shall not be concerned with the finite scheme dependent contact terms that do not affect on-shell quantities but play a crucial role in the consistency of the subtraction procedure [5] and contribute to certain sum rules [28].

Holographic renormalization yields the correct (anomalous) Ward identities, which impose restrictions on the exact one-point function $\langle T^i_j \rangle$ [5, 6],

$$\nabla_i \langle T^i_j \rangle + \nabla_j \hat{\phi} \langle \mathcal{O} \rangle = 0 , \tag{5.6}$$

$$\langle T \rangle + (d - \Delta)\hat{\phi} \langle \mathcal{O} \rangle = \mathcal{A} . \tag{5.7}$$

Here, $\mathcal{A}$ denotes the conformal anomaly, and $\hat{\phi}$ includes also the background source, whereas $\hat{\varphi}$ only denotes the fluctuation source. Hence, we can quite generally restrict the analysis of correlation functions to those containing $\mathcal{O}$ and the traceless transversal part of T^i_j.

Eqs. (5.6) and (5.7) give rise to the following operator identities, which are valid in all (*non-local*) correlation functions at distinct insertion points[5]

$$\partial_i T^i_j = 0 , \qquad T = \beta \mathcal{O} . \tag{5.8}$$

[5]There are also local terms in the correlation functions, which cannot be described by these operator identities. For example, from (5.7) follows

$$\langle T(z)\mathcal{O}(x)\mathcal{O}(y) \rangle = \beta \langle \mathcal{O}(z)\mathcal{O}(x)\mathcal{O}(y) \rangle + [\delta(z-x) + \delta(z-y)] \langle \mathcal{O}(x)\mathcal{O}(y) \rangle + \frac{\partial}{\partial z^i}\delta(z-x)\frac{\partial}{\partial z^i}\delta(z-y) .$$

The last term stems from the anomaly.

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For the GPPZ flow, where $d = 4$, $\Delta = 3$, and $\hat{\hat{\phi}} = \sqrt{3}$, we find $\beta = -\sqrt{3}$. These operator identities can also be understood from our gauge invariant approach. Substituting the decomposition (3.5) into (5.1), we see that ϵ^j is the source of $\partial_i T^i_j$. However, ϵ^j appears in the gauge invariant fields only with an r-derivative, so that its source, which appears in the constant leading term, is irrelevant for the bulk dynamics. Consequently, the dual operator vanishes. Similarly, up to the numerical constant $2(d-1)$, $\hat{h}$ is the source of T, and from the definition of the invariant a we see that the corresponding source is

$$\hat{a} = \hat{\varphi} - \frac{(\Delta - d)}{2(d-1)} \hat{\hat{\phi}} \hat{h} \,, \tag{5.9}$$

from which the second identity in (5.8) follows immediately.

Let us now consider in more detail the exact holographic one-point function $\langle \mathcal{O} \rangle$ in the presence of sources. As we are not interested in finite scheme dependent contact terms, it suffices to calculate the response $\check{a}$. To this end we observe that a satisfies an equation of the form [cf. (4.11)]

$$\left(\tilde{\nabla}^2 - M^2 \right) a = \frac{W_\phi}{W} J_{\tilde{a}} \,, \tag{5.10}$$

where M^2 is an effective mass term, and $\tilde{\nabla}$ now denotes the background covariant derivative. Thus, after defining a covariant Green's function by

$$\left(\tilde{\nabla}^2 - M^2 \right) G(z, z') = \frac{\delta(z - z')}{\sqrt{\tilde{g}(z)}} \,, \tag{5.11}$$

the general solution for a has the form

$$a(z) = \int \mathrm{d}^d y \, K(z, y) \hat{a}(y) + \int \mathrm{d}^{d+1} z' \sqrt{\tilde{g}(z')} G(z, z') \frac{W_\phi(z')}{W(z')} J_{\tilde{a}}(z') \,. \tag{5.12}$$

Here, z is a short notation for the variables (ρ, x), and x and y are boundary coordinates. Notice that the bulk integral is cut off at $\rho' = \varepsilon$, and also that $\rho \geq \varepsilon$, because of the regularization procedure. Moreover, $K(z, y)$ denotes the bulk-to-boundary propagator.

We are interested in the near-boundary behaviour of a, and it is very helpful that in asymptotically AdS spaces the Green's function asymptotically behaves as [29]

$$G(z, z') \approx -\frac{\rho^{\Delta/2}}{2\Delta - d} K(x, z') + \cdots \,. \tag{5.13}$$

Setting also $\rho = \epsilon$, this yields

$$a(\varepsilon, x) \approx \int \mathrm{d}^d y \, K(\varepsilon, x; y) \hat{a}(y) - \frac{\varepsilon^{\Delta/2}}{2\Delta - d} \int_\varepsilon \frac{\mathrm{d}\rho'}{2\rho'} \int \mathrm{d}^d y \, \mathrm{e}^{dA(\rho')} K(x; \rho', y) \frac{W_\phi(\rho')}{W(\rho')} J_{\tilde{a}}(\rho', y) \,. \tag{5.14}$$

It is more useful to consider this expression in momentum space, where we can use momentum conservation in the propagators, $K(\rho, p; q) = K_p(\rho) \delta(p + q)$. We also introduce the bulk-to-boundary propagator for the field $\tilde{a}$, defined by $\tilde{K} = (W/W_\phi) K$. Thus, (5.14) becomes

$$a(\varepsilon, p) \approx \frac{W_\phi(\varepsilon)}{W(\varepsilon)} \tilde{K}_p(\varepsilon) \hat{a}(p) - \frac{\varepsilon^{\Delta/2}}{2\Delta - d} \int_\varepsilon \frac{\mathrm{d}\rho}{2\rho} \mathrm{e}^{dA(\rho)} \tilde{K}_p(\rho) \left[\frac{W_\phi(\rho)}{W(\rho)} \right]^2 J_{\tilde{a}}(\rho, p) \,. \tag{5.15}$$

The two-point function $\langle\mathcal{O}\mathcal{O}\rangle$ can be read off easily from the asymptotic behaviour of the bulk-to-boundary propagator, $\tilde{K}_p$. Were the integral in (5.15) finite for $\varepsilon \to 0$ (removal of the cut-off), it would directly represent the contribution to the response $\breve{a}$ stemming from the interactions, i.e. also the terms needed for the three-point functions. However, one should in general expect the integral to be divergent. This does not cause problems, because the divergences can be understood and even predicted by considering the counter terms in holographic renormalization and are thus easily removed. (Remember that holographic renormalization ensures that the exact one-point function is finite.) The counter terms are obtained quite easily using the Hamilton-Jacobi approach to holographic normalization [6]. In the GPPZ flow, such terms do not appear with the active scalar, because the generic ϕ^4 logarithmic counter term is absent. In contrast, we expect logarithmic divergences from the $(e^i_j)^2$ terms in $J_{\tilde{a}}$, since there is a $\phi^2 R$ logarithmic counter term, and also in the mixed active-inert scalar sector because there are $\phi^2\sigma^2$ logarithmic counter terms [24].

5.2 The Holographic Three-Point Function $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$

In this subsection, we give a general, but formal expression for the three-point function $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$, and we demonstrate that it is Bose symmetric. This is a useful check, because we do not differentiate three times with respect to the source a lagrangian for the field $\tilde{a}$ alone.

The relevant interaction terms are obtained by inserting the linear solutions for b and c from (4.10) into (4.13) dropping d_i and e^i_j. Thus, we find

$$
\begin{aligned}
J_{\tilde{a}} = &\frac{1}{4}\left(\frac{W_\phi}{W}\right)^4 \left[2\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}'\Pi^j_i\tilde{a}'\right) - (\Pi^j_i\tilde{a}')\frac{\partial^i\partial_j}{\Box}\tilde{a}'\right] + 2\left(\frac{W_\phi}{W}\right)^2 \frac{\partial^i\tilde{a}'}{\Box}\partial_i\tilde{a}' + \\
&+\frac{1}{2}\left[\partial_r\left(\frac{W_\phi}{W}\right)^4\right]\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}\Pi^j_i\tilde{a}'\right) + \left[\partial_r\left(\frac{W_\phi}{W}\right)^2\right]\left[\frac{\partial^i\tilde{a}'}{\Box}\partial_i\tilde{a} - \frac{\partial^i}{\Box}\left(\tilde{a}'\partial_i\tilde{a}\right) - \tilde{a}'\tilde{a}\right] - \\
&-\frac{1}{2}\left(\frac{W_\phi}{W}\right)^4 \mathrm{e}^{-2A}\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}\Box\Pi^j_i\tilde{a}\right) + \frac{1}{2}\left[D^2\left(\frac{W_{\phi\phi}}{W} - \frac{W_\phi^2}{W^2}\right)\right]\tilde{a}^2 + \\
&+\left(\frac{W_\phi}{W}\right)^2 \mathrm{e}^{-2A}\left[2\tilde{a}\Box\tilde{a} + \frac{\partial^i}{\Box}\left[(\partial_i\tilde{a})(\Box\tilde{a})\right] - \frac{1}{2}(\partial^i\tilde{a})(\partial_i\tilde{a})\right], \qquad (5.16)
\end{aligned}
$$

where D^2 is the second order differential operator defined in (4.12), and we have abbreviated $\tilde{a}' = \partial_r\tilde{a}$.

The three-point function $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ is formally given by the integral in (5.15), where we should substitute the first order solutions of $\tilde{a}$ into $J_{\tilde{a}}$. However, in this form it is not evident that the final expression will be Bose symmetric. In fact, we expect the integrand to be Bose symmetric up to total derivative terms, which then must vanish in the $\varepsilon \to 0$ limit. In order to facilitate the integrations by parts, it is helpful to perform first a field redefinition that removes from $J_{\tilde{a}}$ the terms of the form $(\tilde{a}')^2$ in the first line of (5.16), [30]. Hence, we perform the replacement

$$
\tilde{a} \to \tilde{a} + \frac{1}{8}\left(\frac{W_\phi}{W}\right)^4\left[2\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}\Pi^j_i\tilde{a}\right) - (\Pi^j_i\tilde{a})\frac{\partial^i\partial_j}{\Box}\tilde{a}\right] + \left(\frac{W_\phi}{W}\right)^2\frac{\partial^i\tilde{a}}{\Box}\partial_i\tilde{a}\,. \qquad (5.17)
$$

After this field redefinition the source in (4.11) becomes

$$
\begin{aligned}
J_{\tilde{a}} = & \frac{1}{2}\left[\partial_r\left(\frac{W_\phi}{W}\right)^4\right]\left[\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}'\frac{\partial^j\partial_i}{\Box}\tilde{a}\right)-\left(\frac{\partial^i\partial_j}{\Box}\tilde{a}'\right)\left(\frac{\partial^j\partial_i}{\Box}\tilde{a}\right)\right] \\
& -\left[\partial_r\left(\frac{W_\phi}{W}\right)^2\right]\left[\frac{\partial^i\tilde{a}'}{\Box}\partial_i\tilde{a}+2\frac{\partial^i\tilde{a}}{\Box}\partial_i\tilde{a}'+\frac{\partial^i}{\Box}\left(\tilde{a}'\partial_i\tilde{a}\right)+\tilde{a}\tilde{a}'\right] \\
& -\frac{1}{2}\left(\frac{W_\phi}{W}\right)^4 \mathrm{e}^{-2A}\Bigg[\tilde{a}\Box\tilde{a}-\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}\partial_i\partial^j\tilde{a}\right)+\frac{1}{2}(\partial_i\tilde{a})(\partial^i\tilde{a})-\frac{\partial^i\partial_j}{\Box}\left((\partial_k\tilde{a})\frac{\partial_i\partial^j\partial^k}{\Box}\tilde{a}\right)+ \\
& \qquad +\frac{1}{2}\left(\frac{\partial^i\partial^j\partial^k}{\Box}\tilde{a}\right)\left(\frac{\partial_i\partial_j\partial_k}{\Box}\tilde{a}\right)\Bigg]+ \\
& +\left(\frac{W_\phi}{W}\right)^2 \mathrm{e}^{-2A}\left\{2\tilde{a}\Box\tilde{a}+\frac{\partial^i}{\Box}\left[(\partial_i\tilde{a})(\Box\tilde{a})\right]-\frac{1}{2}(\partial^i\tilde{a})(\partial_i\tilde{a})-2\left(\frac{\partial^i\partial_j}{\Box}\tilde{a}\right)\left(\partial_i\partial^j\tilde{a}\right)\right\}+ \\
& +\frac{1}{2}\left[D^2\left(\frac{W_{\phi\phi}}{W}-\frac{W_\phi^2}{W^2}\right)\right]\tilde{a}^2-\left[D^2\left(\frac{W_\phi}{W}\right)^2\right]\frac{\partial^i\tilde{a}}{\Box}\partial_i\tilde{a}- \\
& -\frac{1}{8}\left[D^2\left(\frac{W_\phi}{W}\right)^4\right]\left[\tilde{a}^2-2\frac{\partial^i\partial_j}{\Box}\left(\tilde{a}\frac{\partial^j\partial_i}{\Box}\tilde{a}\right)+\left(\frac{\partial^i\partial_j}{\Box}\tilde{a}\right)\left(\frac{\partial^j\partial_i}{\Box}\tilde{a}\right)\right].
\end{aligned}
\tag{5.18}
$$

The three-point function of the active scalar is given by the integral in (5.15),[6]

$$
(2\Delta-d)\check{a}^{(2)}(p) = -\int_0^\infty \mathrm{d}r\,\mathrm{e}^{dA}\left(\frac{W_\phi}{W}\right)^2 \tilde{K}_p(r)J_{\tilde{a}}(r,p)\,. \tag{5.19}
$$

Differentiating $\check{a}^{(2)}$ twice with respect to $\hat{a}(-p)$ yields the three-point function $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$. In order to do this, note that $J_{\tilde{a}}(r,p)$ is of the form

$$
J_{\tilde{a}}(r,p) = \int \mathrm{d}p_2\,\mathrm{d}p_3\,\delta(p+p_2+p_3)\mathcal{X}(p,-p_2,-p_3)\tilde{K}_2\tilde{K}_3\hat{a}(-p_2)\hat{a}(-p_3)\,, \tag{5.20}
$$

where $\tilde{K}_2$ and $\tilde{K}_3$ stand for the bulk-to-boundary propagators $\tilde{K}_{p_2}(r)$ and $\tilde{K}_{p_3}(r)$, respectively, and the operator $\mathcal{X}$, which includes derivatives with respect to r acting on $\tilde{K}_2$ and $\tilde{K}_3$, can be read off from (5.18). It is important to notice the minus signs of the momenta p_2 and p_3. Thus, the three-point function we are seeking is

$$
\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = -\delta(p_1+p_2+p_3)\int_0^\infty \mathrm{d}r\,\mathrm{e}^{dA}\left(\frac{W_\phi}{W}\right)^2 \mathcal{X}_{123}\tilde{K}_1\tilde{K}_2\tilde{K}_3\,, \tag{5.21}
$$

where

$$
\mathcal{X}_{123} = \mathcal{X}(p_1,-p_2,-p_3)+\mathcal{X}(p_1,-p_3,-p_2)\,. \tag{5.22}
$$

[6] We might need to subtract divergences, if holographic renormalization predicts them by the presence of the appropriate counter terms.

After reading off $\mathcal{X}$ from (5.18) and using momentum conservation, $p_1 + p_2 + p_3 = 0$, we find

$$
\begin{aligned}
\mathcal{X}_{123} = {} & \frac{1}{2}\left[\partial_r\left(\frac{W_\phi}{W}\right)^4\right]\left[\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2}\partial_3 + \frac{(p_1\cdot p_3)^2}{p_1^2p_3^2}\partial_2 + \frac{(p_2\cdot p_3)^2}{p_2^2p_3^2}\partial_1 - \right. \\
& \qquad\qquad\qquad \left. - \frac{(p_2\cdot p_3)^2}{p_2^2p_3^2}(\partial_1+\partial_2+\partial_3)\right] + \\
& + \left[\partial_r\left(\frac{W_\phi}{W}\right)^2\right]\left[p_2\cdot p_3\left(\frac{1}{p_2^2}+\frac{1}{p_3^2}\right)\partial_1 + p_1\cdot p_3\left(\frac{1}{p_1^2}+\frac{1}{p_3^2}\right)\partial_2 + \right. \\
& \left. + p_1\cdot p_2\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right)\partial_3 - p_2\cdot p_3\left(\frac{1}{p_2^2}+\frac{1}{p_3^2}\right)(\partial_1+\partial_2+\partial_3)\right] + \\
& + \frac{1}{2}\left(\frac{W_\phi}{W}\right)^4 \mathrm{e}^{-2A}\left[\frac{1}{2}(p_1^2+p_2^2+p_3^2) + \frac{(p_1\cdot p_2)^3}{p_1^2p_2^2} + \frac{(p_1\cdot p_3)^3}{p_1^2p_3^2} + \frac{(p_2\cdot p_3)^3}{p_2^2p_3^2}\right] + \\
& + \left(\frac{W_\phi}{W}\right)^2 \mathrm{e}^{-2A}\left[(p_2\cdot p_3)^2\left(\frac{1}{p_2^2}+\frac{1}{p_3^2}\right) + (p_1\cdot p_3)^2\left(\frac{1}{p_1^2}+\frac{1}{p_3^2}\right) + \right. \\
& \qquad\qquad \left. + (p_1\cdot p_2)^2\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right) - 2(p_1^2+p_2^2+p_3^2)\right] + \\
& + \frac{1}{4}\left[D^2\left(\frac{W_\phi}{W}\right)^4\right]\left[\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2} + \frac{(p_1\cdot p_3)^2}{p_1^2p_3^2} - \frac{(p_2\cdot p_3)^2}{p_2^2p_3^2}\right] + \\
& + \left\{D^2\left[\frac{W_{\phi\phi}}{W} - \left(\frac{W_\phi}{W}\right)^2 - \frac{1}{4}\left(\frac{W_\phi}{W}\right)^4\right]\right\} - \left[D^2\left(\frac{W_\phi}{W}\right)^2\right] p_2\cdot p_3\left(\frac{1}{p_2^2}+\frac{1}{p_3^2}\right).
\end{aligned}
\tag{5.23}
$$

The symbols ∂_n, $n = 1, 2, 3$, denote the derivative with respect to r acting on the bulk-to-boundary propagator $\tilde{K}_n$. As

$$
(\partial_1+\partial_2+\partial_3)(\tilde{K}_1\tilde{K}_2\tilde{K}_3) = \partial_r(\tilde{K}_1\tilde{K}_2\tilde{K}_3)\,, \tag{5.24}
$$

we can integrate these two terms of (5.23) by parts in the integral in (5.21) exploiting also the identity

$$
\partial_r\left[\mathrm{e}^{dA}\left(\frac{W_\phi}{W}\right)^2\partial_r\right] = \mathrm{e}^{dA}\left(\frac{W_\phi}{W}\right)^2 D^2\,. \tag{5.25}
$$

Hence, the last term in (5.23) is cancelled, and the minus sign of the last term on the penultimate line is reversed, rendering the final result totally symmetric in the indices 1, 2 and 3. This is the proof of Bose symmetry we wanted. Thus, the final result is

$$
\begin{aligned}
\mathcal{X}_{123} = {} & \frac{1}{2}\left[\partial_r\left(\frac{W_\phi}{W}\right)^4\right]\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2}\partial_3 + \left[\partial_r\left(\frac{W_\phi}{W}\right)^2\right] p_1\cdot p_2\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right)\partial_3 + \\
& + \frac{1}{2}\left(\frac{W_\phi}{W}\right)^4 \mathrm{e}^{-2A}\left[\frac{1}{2}p_1^2 + \frac{(p_1\cdot p_2)^3}{p_1^2p_2^2}\right] + \\
& + \left(\frac{W_\phi}{W}\right)^2 \mathrm{e}^{-2A}\left[(p_1\cdot p_2)^2\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right) - 2p_1^2\right] +
\end{aligned}
\tag{5.26}
$$

$$+\frac{1}{4}\left[D^2\left(\frac{W_\phi}{W}\right)^4\right]\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2}+\frac{1}{3}\left\{D^2\left[\frac{W_{\phi\phi}}{W}-\left(\frac{W_\phi}{W}\right)^2-\frac{1}{4}\left(\frac{W_\phi}{W}\right)^4\right]\right\}+\text{cyclic}\,.$$

In the above analysis we have not really been careful dropping the boundary terms in the integration by parts in the step from (5.23) to (5.26) and using the field redefinition (5.17), both of which might contribute terms to the three-point function that are not Bose symmetric. Thus, it remains to check that these contributions either cancel or vanish in the limit $\varepsilon \to 0$. We shall show this explicitly for the GPPZ flow in the next subsection, but the argument extends easily to the general case.

5.3 Correlation functions in the GPPZ Flow

We shall now apply our results to the GPPZ flow, where it is conventional to use the variable $u = 1-\rho = 1-\mathrm{e}^{-2r}$. Useful identities are listed in appendix A. For completeness, we also calculate the two-point function using our simpler linear equation, although the result has been known for some time. We would like to mention that we have performed the calculations for dimensionless p. In order to restore the proper dimensions, one must replace p by pL everywhere, where L is the radius of curvature of the asymptotic AdS region ($L^4 = 4\pi g_s N\alpha'^2$), so that $\mathcal{O}(p) \to \mathcal{O}(pL) = \mathcal{O}(p)/L$. ($\mathcal{O}(p)$ has dimension -1, corresponding to $\mathcal{O}(x)$ of dimension 3.) Furthermore, the results should be multiplied by the numerical factor $[N^2/(2\pi^2)] \times (2\pi)^4$, where the $(2\pi)^4$ stems from our convention for the δ-function in momentum space.

The equation of motion (4.11) for $\tilde{a}$ becomes (in momentum space)

$$\left[u(1-u)\partial_u^2+(2-2u)\partial_u-\frac{p^2}{4}\right]\tilde{a}=\frac{u}{4(1-u)}J_{\tilde{a}}\,. \tag{5.27}$$

The associated homogeneous equation is a hypergeometric equation, whose solution, which is regular for $u=0$, is readily found. We have to be somewhat careful with the normalization, because the expansion (5.3) should hold also for $K_p = (W_\phi/W)\tilde{K}_p$ (in particular with a factor 1 in the leading term). Hence, we find the bulk-to-boundary propagator

$$\tilde{K}_p(u)=\frac{\sqrt{3}}{2}\Gamma\left(\frac{3+\alpha}{2}\right)\Gamma\left(\frac{3-\alpha}{2}\right)\mathrm{F}\left(\frac{1+\alpha}{2},\frac{1-\alpha}{2};2;u\right), \tag{5.28}$$

with $\alpha=\sqrt{1-p^2}$. Its asymptotic behaviour is given by [31]

$$\tilde{K}_p(u)\approx\frac{\sqrt{3}}{2}\left[1+\frac{p^2}{4}(1-u)\ln(1-u)+\frac{1}{2}H(p)(1-u)+\cdots\right]. \tag{5.29}$$

The function $H(p)$, which is related to the two-point function by

$$\langle\mathcal{O}(p)\mathcal{O}(q)\rangle=\delta(p+q)H(p)\,, \tag{5.30}$$

is given by

$$H(p)=\frac{p^2}{2}\left[\psi\left(\frac{3+\alpha}{2}\right)+\psi\left(\frac{3-\alpha}{2}\right)-\psi(2)-\psi(1)\right]. \tag{5.31}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. The spectrum of poles becomes clear after rewriting H in a series representation using the formula [32]

$$\psi(x) - \psi(y) = \sum_{k=0}^{\infty} \left(\frac{1}{y+k} - \frac{1}{x+k} \right). \tag{5.32}$$

This yields

$$H(p) = \frac{p^4}{2} \sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)[4k(k+1)+p^2]}. \tag{5.33}$$

Thus, we find particles with the masses $m^2 = 4k(k+1)$, $k = 1, 2, 3, \ldots$. The residues at the poles, which represent the decay constants [33], are

$$|f_k|^2 = 8k(k+1)(2k+1). \tag{5.34}$$

Let us now consider the three-point function. In the case of the GPPZ flow, we obtain from (5.21) and (5.26)

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = -\frac{2}{27} \delta(p_1 + p_2 + p_3) \int_0^1 \mathrm{d}u \, \mathcal{Y}_{123} \tilde{K}_1 \tilde{K}_2 \tilde{K}_3 , \tag{5.35}$$

where

$$\begin{aligned}
\mathcal{Y}_{123} &= \frac{9u^2}{(1-u)^2} \mathcal{X}_{123} \\
&= -64u^2(1-u) \frac{(p_1 \cdot p_2)^2}{p_1^2 p_2^2} \partial_3 - 48u^2 p_1 \cdot p_2 \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} \right) \partial_3 + \\
&\quad +8u(1-u) \left[\frac{1}{4}(p_1^2 + p_2^2) + \frac{(p_1 \cdot p_2)^3}{p_1^2 p_2^2} \right] + 12u \left[(p_1 \cdot p_2)^2 \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} \right) - (p_1^2 + p_2^2) \right] + \\
&\quad +32u(3u-2) \frac{(p_1 \cdot p_2)^2}{p_1^2 p_2^2} + 16u \left[2(1-u) + \frac{1}{3} \right] + \text{cyclic}.
\end{aligned} \tag{5.36}$$

Here, ∂_n denotes the derivative with respect to u acting on $\tilde{K}_n$. It is reassuring that $\mathcal{Y}_{123}$ is finite for $u \to 1$.

As anticipated in subsection 5.2, we need to check that the field redefinition and the integration by parts used to obtain the final result do not spoil it. On the one hand, the total derivative terms discarded in the step from (5.23) to (5.26) are

$$\mathrm{e}^{4A} \left(\frac{W_\phi}{W} \right)^2 \tilde{K}_1 \tilde{K}_2 \tilde{K}_3 \delta(p_1 + p_2 + p_3) \partial_r \left[\frac{1}{2} \left(\frac{W_\phi}{W} \right)^4 \frac{(p_2 \cdot p_3)^2}{p_2^2 p_3^2} + \left(\frac{W_\phi}{W} \right)^2 p_2 \cdot p_3 \left(\frac{1}{p_2^2} + \frac{1}{p_3^2} \right) \right], \tag{5.37}$$

where we must take the limit $r \to \infty$. After going to the u variable using the relations of appendix A, as well as $\tilde{K}_n = \sqrt{3}/2 + \cdots$ from (5.29), we obtain

$$-\frac{4}{\sqrt{3}} p_2 \cdot p_3 \left(\frac{1}{p_2^2} + \frac{1}{p_3^2} \right) \delta(p_1 + p_2 + p_3). \tag{5.38}$$

On the other hand, the contribution from the field redefinition is found by differentiating the quadratic terms on the right hand side of (5.17) twice with respect to the source $\hat{a}(-p)$. This yields

$$
\begin{aligned}
2\Bigg[\frac{1}{4}\left(\frac{W_\phi}{W}\right)^4 &\left(1-\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2}-\frac{(p_1\cdot p_3)^2}{p_1^2p_3^2}+\frac{(p_2\cdot p_3)^2}{p_2^2p_3^2}\right)+\\
+\left(\frac{W_\phi}{W}\right)^2 & p_2\cdot p_3\left(\frac{1}{p_2^2}+\frac{1}{p_3^2}\right)\Bigg]\tilde{K}_2\tilde{K}_3\delta(p_1+p_2+p_3)= \\
&=\frac{W}{W_\phi}\left[(1-u)^{3/2}\frac{4}{\sqrt{3}}p_2\cdot p_3\left(\frac{1}{p_2^2}+\frac{1}{p_3^2}\right)+\mathcal{O}((1-u)^{5/2})\right]\delta(p_1+p_2+p_3)\,.
\end{aligned}
\tag{5.39}
$$

We have factored out W/W_ϕ in order to put in evidence the contribution to the response $\check{a}$. Obviously, it cancels the total derivative term (5.38).

In order to find the irreducible trilinear couplings of on-shell states, we need to amputate the legs of the three-point function [33]. This is done best by rewriting the bulk-to-boundary propagator (5.28) around an on-shell pole as

$$
\begin{aligned}
\tilde{K}_p(u) &= \frac{\sqrt{3}}{2}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)(1-u)\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{F}\left(\frac{1+\alpha}{2},\frac{1-\alpha}{2};1;u\right)\\
&= \frac{|f_k|}{p^2+4k(k+1)}\left[\sqrt{\frac{3(2k+1)}{2k(k+1)}}(1-u)\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{P}_k(2u-1)\right]+\text{regular}\,,
\end{aligned}
\tag{5.40}
$$

where f_k is the decay constant defined in (5.34), and P_k denotes the Legendre polynomial of degree k, which satisfies the useful formula

$$
\mathrm{P}_k(2u-1)=\sum_{n=0}^{k}\binom{k}{n}^2 u^n(u-1)^{k-n}\,.
\tag{5.41}
$$

Thus, we can write the three-point function (5.35) close to the on-shell poles as

$$
\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle=\delta(p_1+p_2+p_3)\prod_{i=1}^{3}\left[\frac{|f_{k_i}|}{p_i^2+4k_i(k_i+1)}\right]\mathcal{M}+\text{regular}\,,
\tag{5.42}
$$

where $\mathcal{M}$ denotes the amplitude

$$
\mathcal{M}=-\frac{1}{9}\sqrt{\frac{3}{2}}\prod_{i=1}^{3}\sqrt{\frac{2k_i+1}{k_i(k_i+1)}}\int_0^1\mathrm{d}u\,\mathcal{Y}_{123}F_1F_2F_3\,,
\tag{5.43}
$$

with

$$
F_i(u)=(1-u)\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{P}_{k_i}(2u-1)\,.
\tag{5.44}
$$

Moreover, one should substitute $p_1\cdot p_2=(p_3^2-p_1^2-p_2^2)/2$ and cyclic, with $p_i^2=-4k_i(k_i+1)$, into (5.36).

The integral in (5.43) is elementary for all k_i, but, contrary to the simpler case of the trilinear couplings of two inert scalars to the active scalar [24], the final result is not particularly illuminating, and we do not display it here. We would only like to mention that it satisfies a sort of 'triangular inequality' as a consequence of the orthogonality of the Legendre polynomials $\mathrm{P}_{k_i}(2u-1)$ in the interval $u \in [0,1]$. The $1/p^2$ terms in the operator $\mathcal{Y}_{123}$ defined in (5.36) are not dangerous, because the spectrum contains no massless states. Finally, (5.42) has the correct (mass) dimension and large-N suppression.

6. Conclusions and outlook

In this paper, we have developed and used a gauge invariant approach for the analysis of the equations governing the dynamics of active scalar fluctuations in RG flows. This approach has enabled us to arrive at a second order ODE for the active scalar in a rather simple fashion, which in turn made it possible to use the Green's function method to deal with the quadratic interaction terms. Thus, we have taken another step beyond the tradition and filled a gap for active scalar operators, whose three-point function has not been analyzed in [24].

As an application, we derived an explicitly Bose symmetric formula for the three-point function $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$. To arrive at the Bose symmetric expression a field redefinition removing the interaction terms with two r-derivatives was instrumental. It was then necessary to integrate by parts the integral for the three-point function, with the boundary terms cancelling the contributions from the field redefinition. Other three-point functions can be calculated in the same fashion, which we leave as a future project. We anticipate that the use of the gauge invariant approach simplifies matters significantly. It would also be interesting to cross-check three-point functions like $\langle T^i_j \mathcal{O}\mathcal{O} \rangle$ calculating them in two different ways (in this case, using the Green's function for a and e^i_j).

As mentioned after (5.5), we have not been concerned about the scheme dependent local terms. They are interesting, because some central charges and sum rules may depend on them [34, 28, 35, 36]. Moreover, in the supersymmetric renormalization scheme the finite counter terms without derivatives are uniquely determined. It is not clear to us whether it also determines the other finite counter terms. Hence, a deeper investigation of this matter is advisable.

We have applied our final result to the GPPZ flow, which is particularly simple. As the bulk-to-boundary propagator of on-shell states is a polynomial in this case, the final integral can be carried out explicitly, although it is very difficult to give a general formula for arbitrary masses. This can be used to extract, e.g. the decay rates for the various decay channels. A thorough analysis involving also the states arising from the inert scalars and the energy-momentum tensor looks feasible. Moreover, we leave it to the future to apply the result to the Coulomb branch flow.

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A. Useful relations for the GPPZ flow

We summarize here a number of relations for the GPPZ background. For simplicity, we set the asymptotically AdS length scale to unity, i.e. $L = 1$.

The superpotential for the active scalar is

$$W = -\frac{3}{4}\left(\cosh\frac{2\phi}{\sqrt{3}} + 1\right). \tag{A.1}$$

This potential leads to a significant simplification of its various derivatives, because of the identities

$$W_{\phi\phi} = \frac{4}{3}W + 1\,, \qquad \frac{W_\phi^2}{W} = \frac{4}{3}W + 2\,. \tag{A.2}$$

Integrating the background equations (1.6) yields

$$\mathrm{e}^{2\bar{\phi}/\sqrt{3}} = \frac{1+\mathrm{e}^{-r}}{1-\mathrm{e}^{-r}}\,, \qquad \mathrm{e}^{2A} = \mathrm{e}^{2r} - 1\,. \tag{A.3}$$

From (A.3) we easily find the background source,

$$\hat{\bar{\phi}} = \sqrt{3}\,. \tag{A.4}$$

It is useful to introduce the variable

$$u = 1 - \mathrm{e}^{-2r}\,, \tag{A.5}$$

in terms of which the following relations hold,

$$\begin{aligned} \frac{du}{dr} &= 2(1-u)\,, \qquad & \mathrm{e}^{-2A} &= \frac{1-u}{u}\,, \\ W &= -\frac{3}{2u}\,, \qquad & W_\phi &= -\sqrt{3}\frac{\sqrt{1-u}}{u}\,. \end{aligned} \tag{A.6}$$

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